Problem 3.

Duck's Cone

The problem

If one looks at the wave pattern produced by a duck paddling across a pond, this reminds one of Mach's cone. On what parameters does the pattern depend?

Outline

- Real duck cones
- Our duck simulation apparatus
- Basic properties of duck waves
- Advanced theory
- Comparison of theory and measurements

Real duck cones



Velocity estimate:

0.7 m/s

Real duck cones



Velocity estimate:

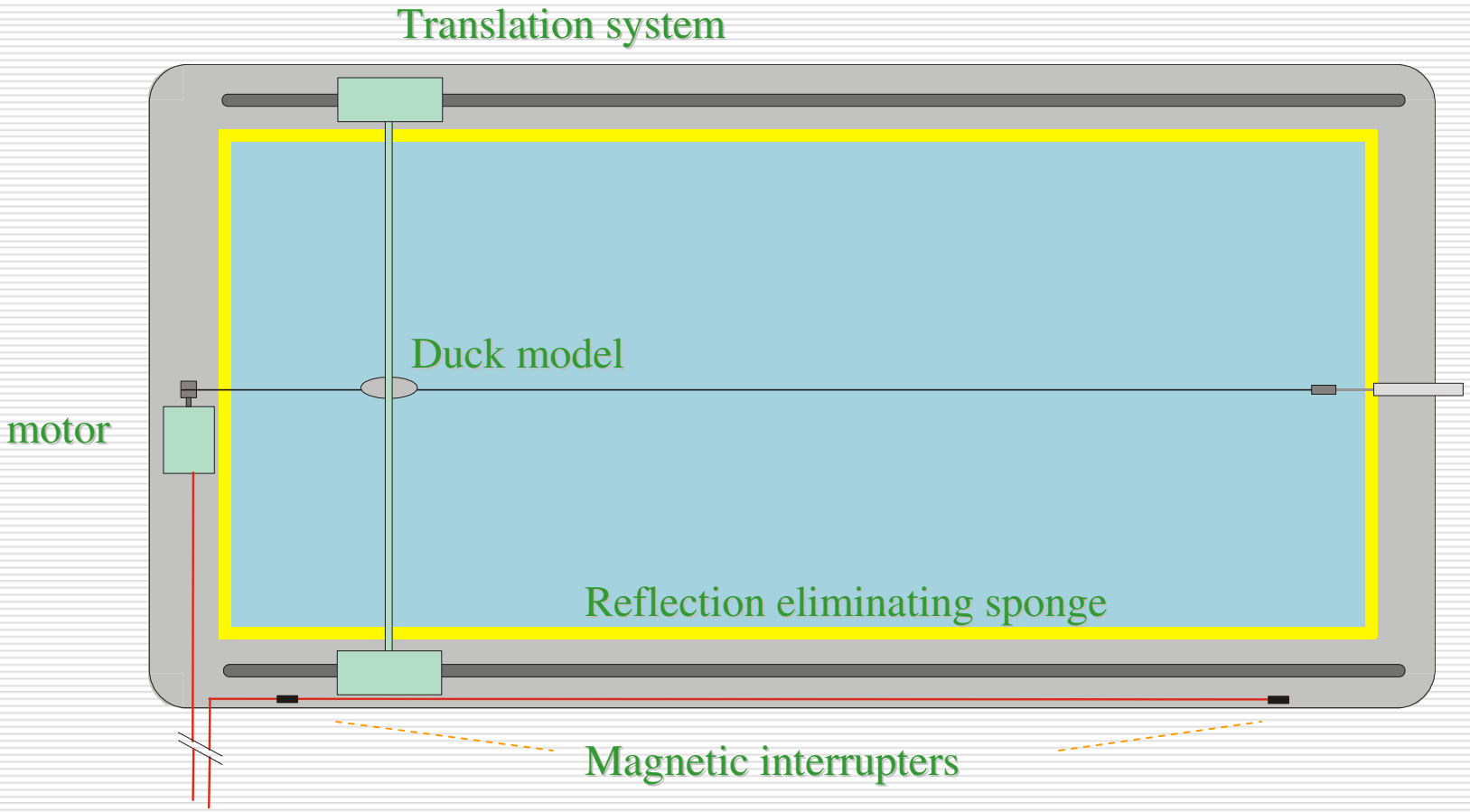
0.4 m/s

Wave pool

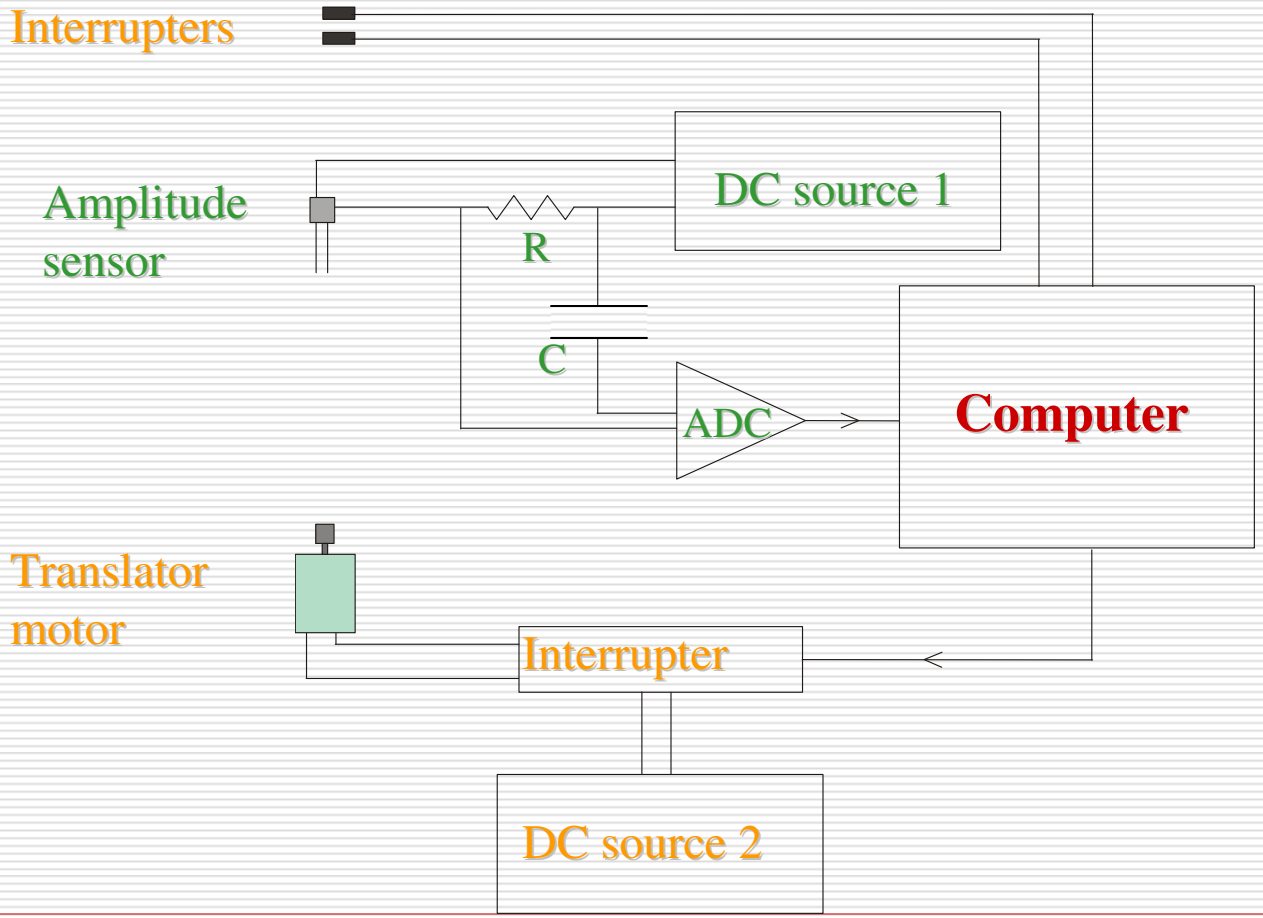
- Our pool – simulation of swimming ducks
- Duck model translation system
- Velocity measurement – magnetic interrupters
- Lighting system – photographing the waves
- Resistance wave amplitude sensor

Pool length [m]	Pool width [m]	Water depth [m]
1.50	0.70	0.23

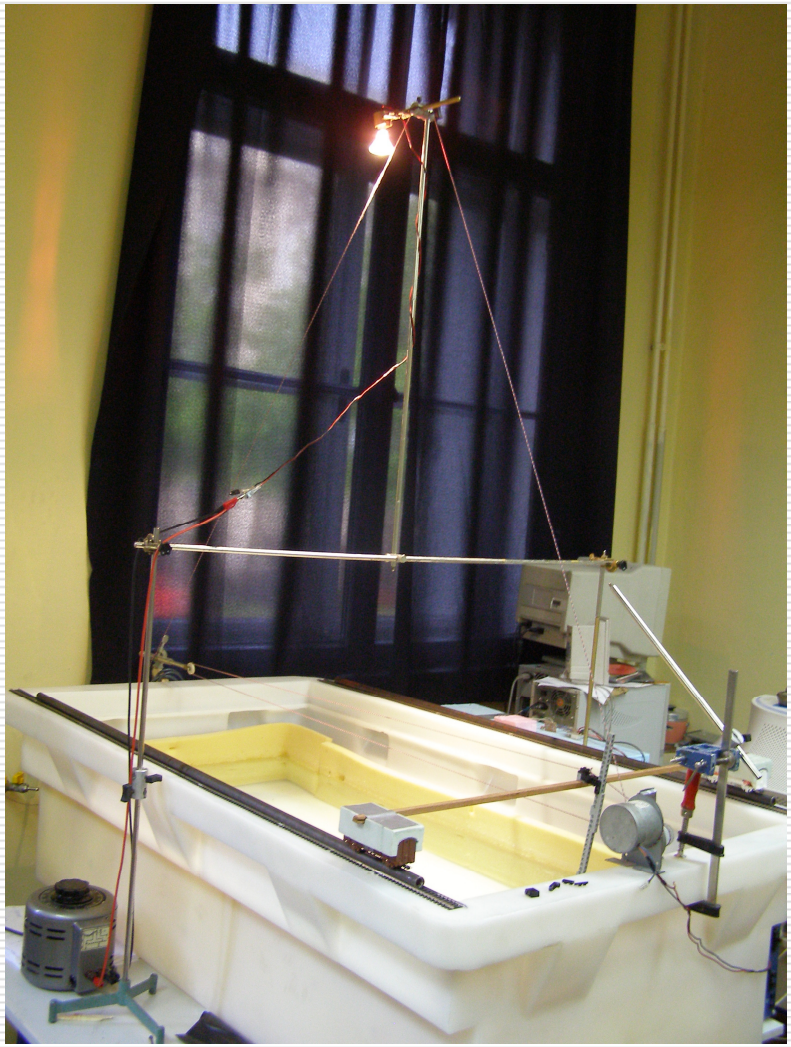
Wave pool cont.

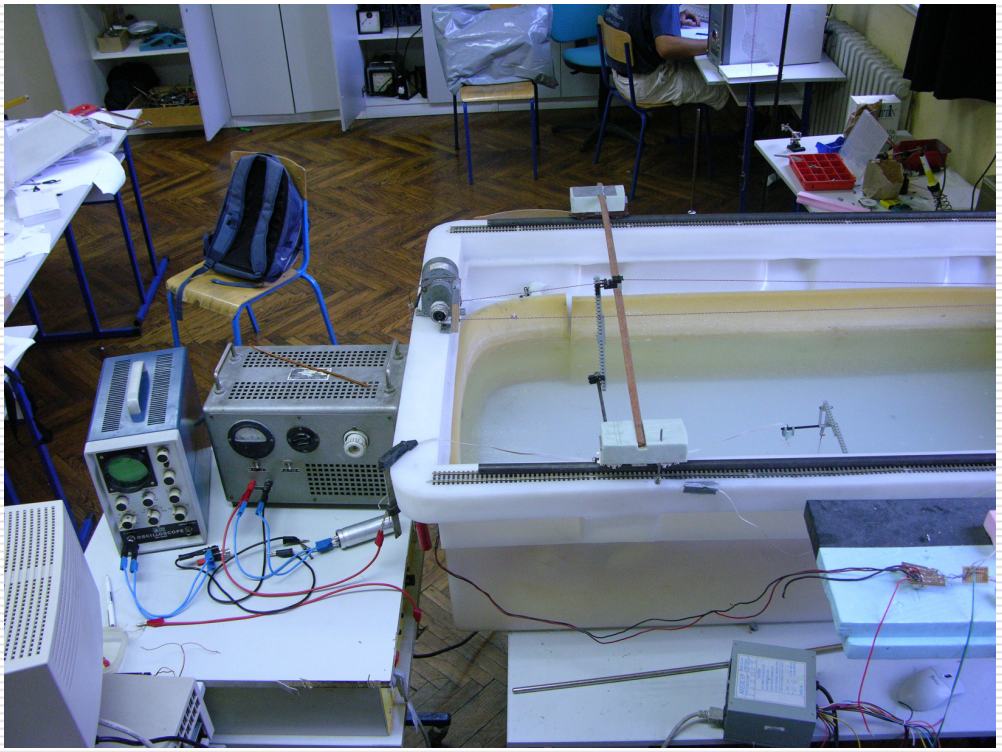


Wave pool cont.

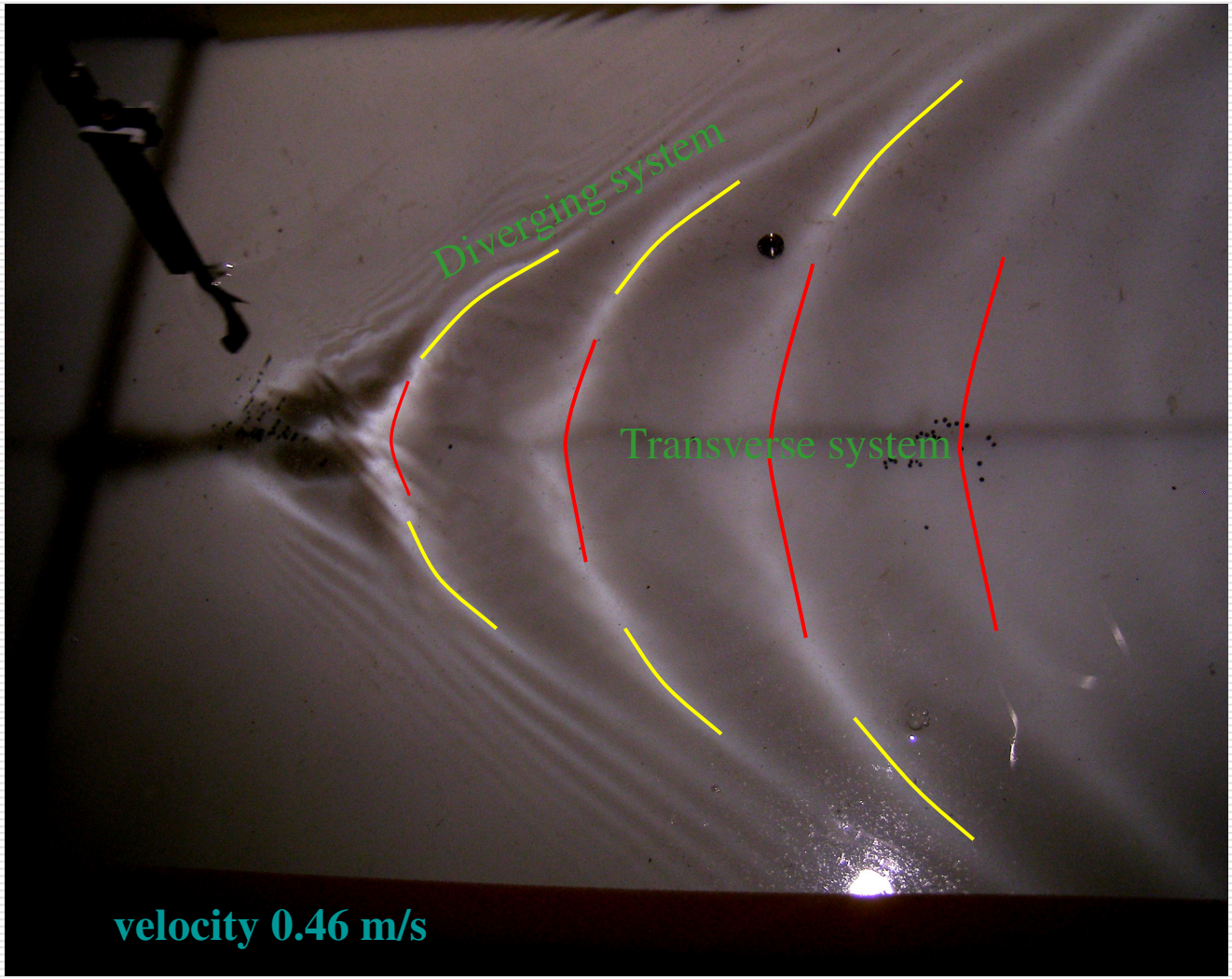


Wave pool cont.





Basic pattern properties



- Feathered appearance
- Two wave systems

Basic pattern properties cont.

- Duck – moving gravitational water wave spectrum emitter
- Distinctive pattern – interference
- Dispersive waves – difference from Mach cone!
- Main parameters:
  - Velocity
  - Source shape
  - (Water depth)

Basic pattern properties cont.

- Quantities of interest:
  - ″cone″ angle
  - Specific wavelengths
  - Amplitudes & their distribution
- Basic theory – phase related, geometrical
- Dispersivity
- Advanced theory – wave equation solving

Basic pattern properties

- Gravitational water waves in deep water – dispersional:

$$c^2 = gk$$

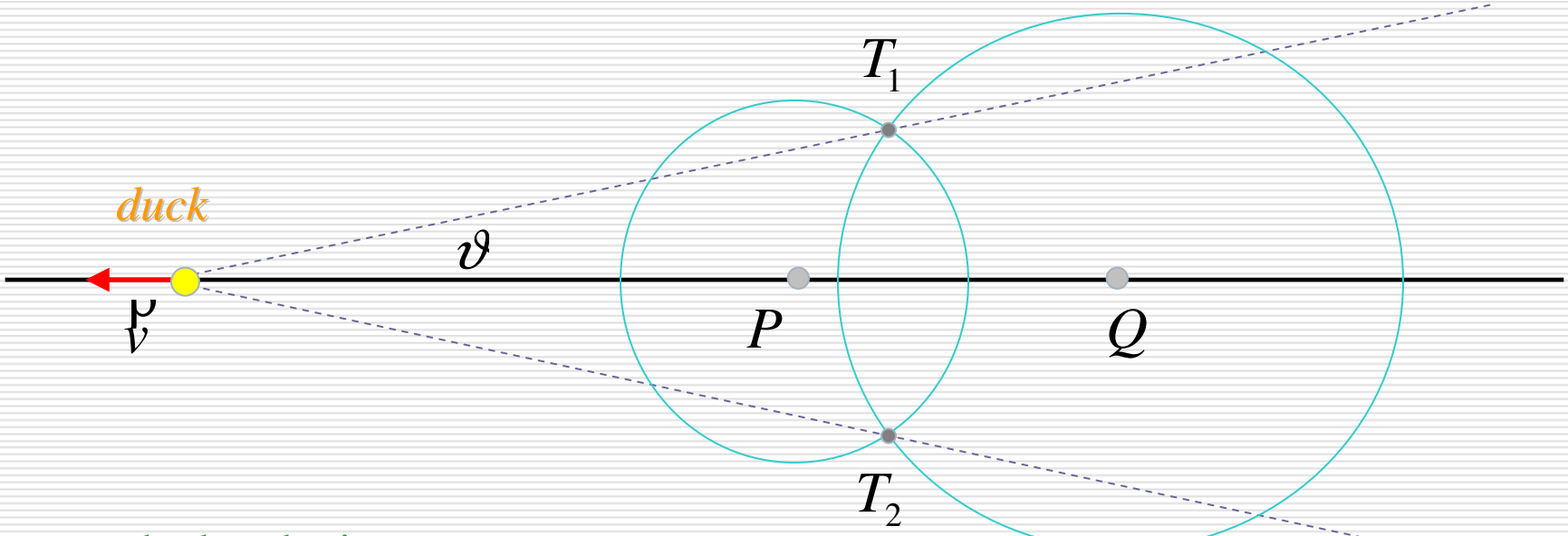
c – phase velocity

g – acceleration of gravity

k – wavenumber value

- Duck – localized source in uniform motion
- Creates surface elevation by moving through water
- Emits the whole spectrum!

Basic pattern properties cont.



v – duck velocity

P,Q – picked emission points

T_1,T_2 – points of interference of distinct wave packets

θ – respective extrema angle

Basic pattern properties cont.

- In duck system – circular symmetry
- Duck emission:

$$\psi \sim \int_0^\infty a(k)\cos k(r' - ct)dk$$

$$\Rightarrow \quad \psi = A(r')\sin\frac{gt^2}{4r'}$$

ψ – surface elevation
$a(k)$ – spectral distribution
r' – radial distance from duck
t – time
$A(r')$ – amplitude function
k – wavenumber value

Basic pattern properties cont.

- In water system – linear coordinate transformations yield phase
- Points of time-stationary phase – interference extrema:

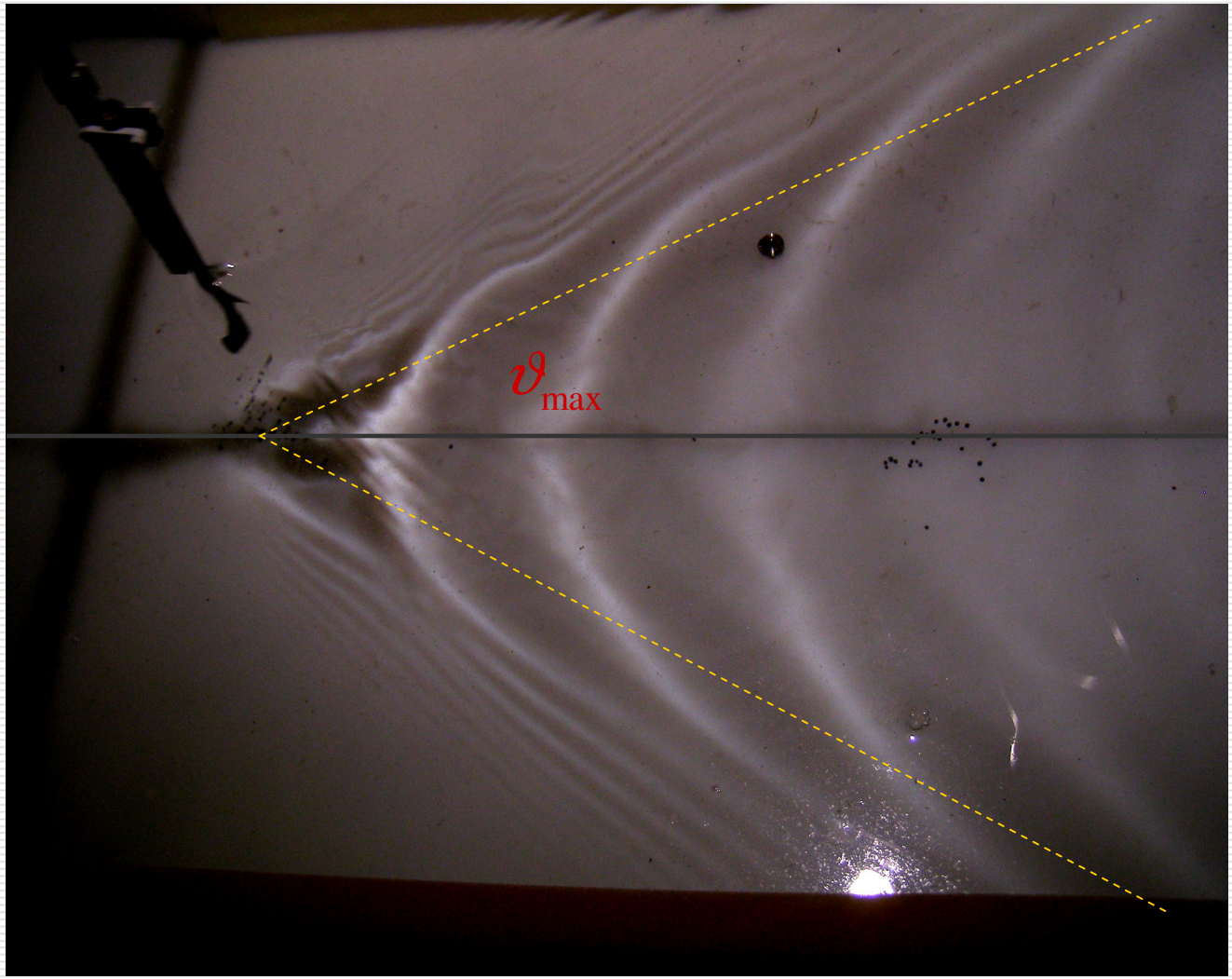
$$\frac{d\phi}{dt} = 0 \quad \Rightarrow \quad \psi = extremum \quad \text{– isophasal curves}$$

$$\Rightarrow \quad \boxed{\cos^2 \vartheta \le \frac{8}{9}}$$

$$\vartheta \le 19.5°$$

ϕ – phase
ψ – surface elevation
θ_{max} – interference angle

Basic pattern properties cont.



- Maximum angle independent of velocity!
- Experimental:

$\vartheta_{max} = 19 \pm 1°$

Velocity range:

0.25 – 0.7 m/s

Basic pattern properties cont.

- Transverse system – traveling at duck velocity

$\Rightarrow$

$$\lambda_T = \frac{2\pi}{g} v^2$$

λ_T – transverse wavelength

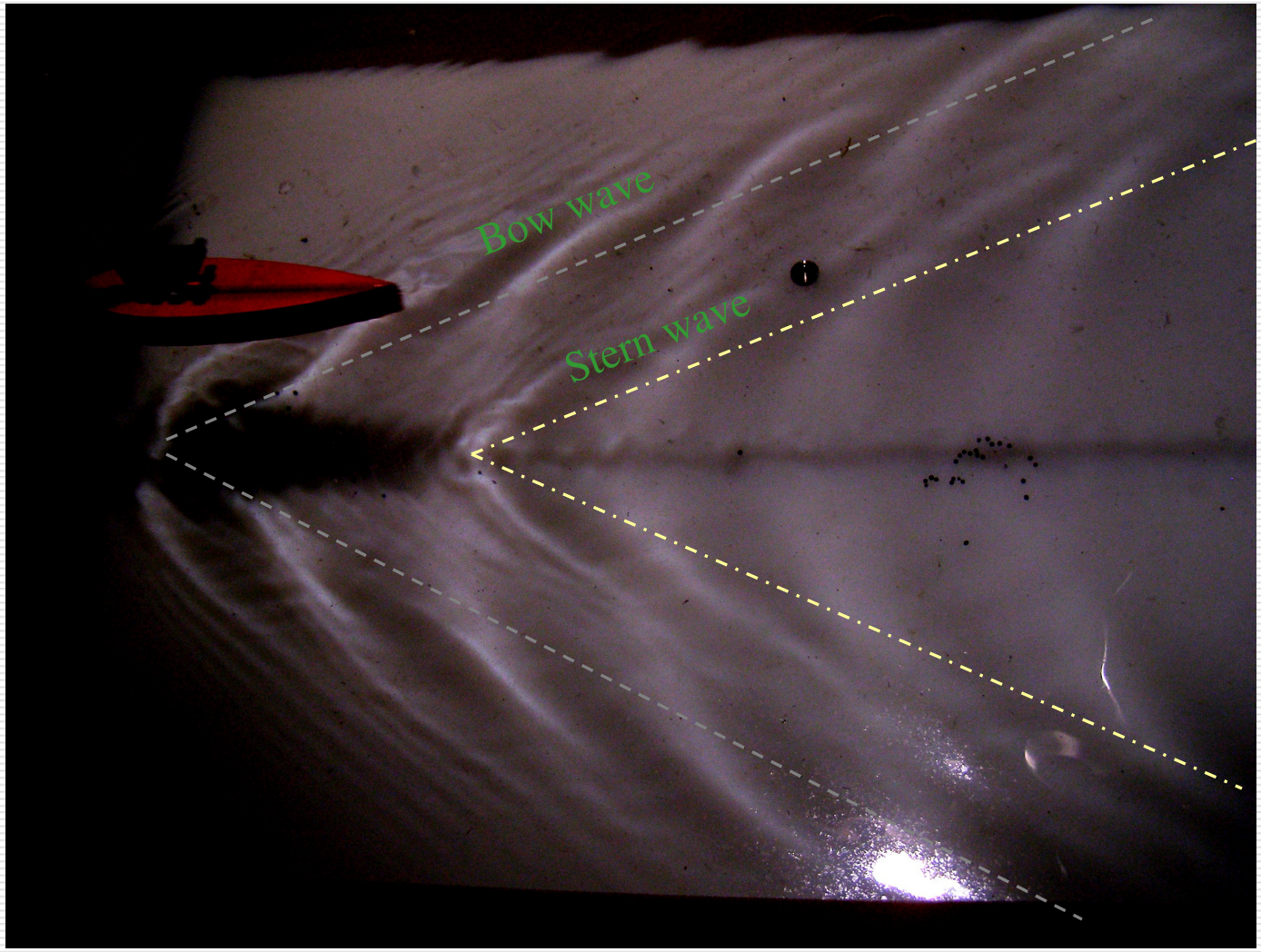
v – duck velocity

- Diverging waves – growing wavelength as well
- Dependence of pattern on source shape – causes exact amplitude distribution

Basic pattern properties cont.

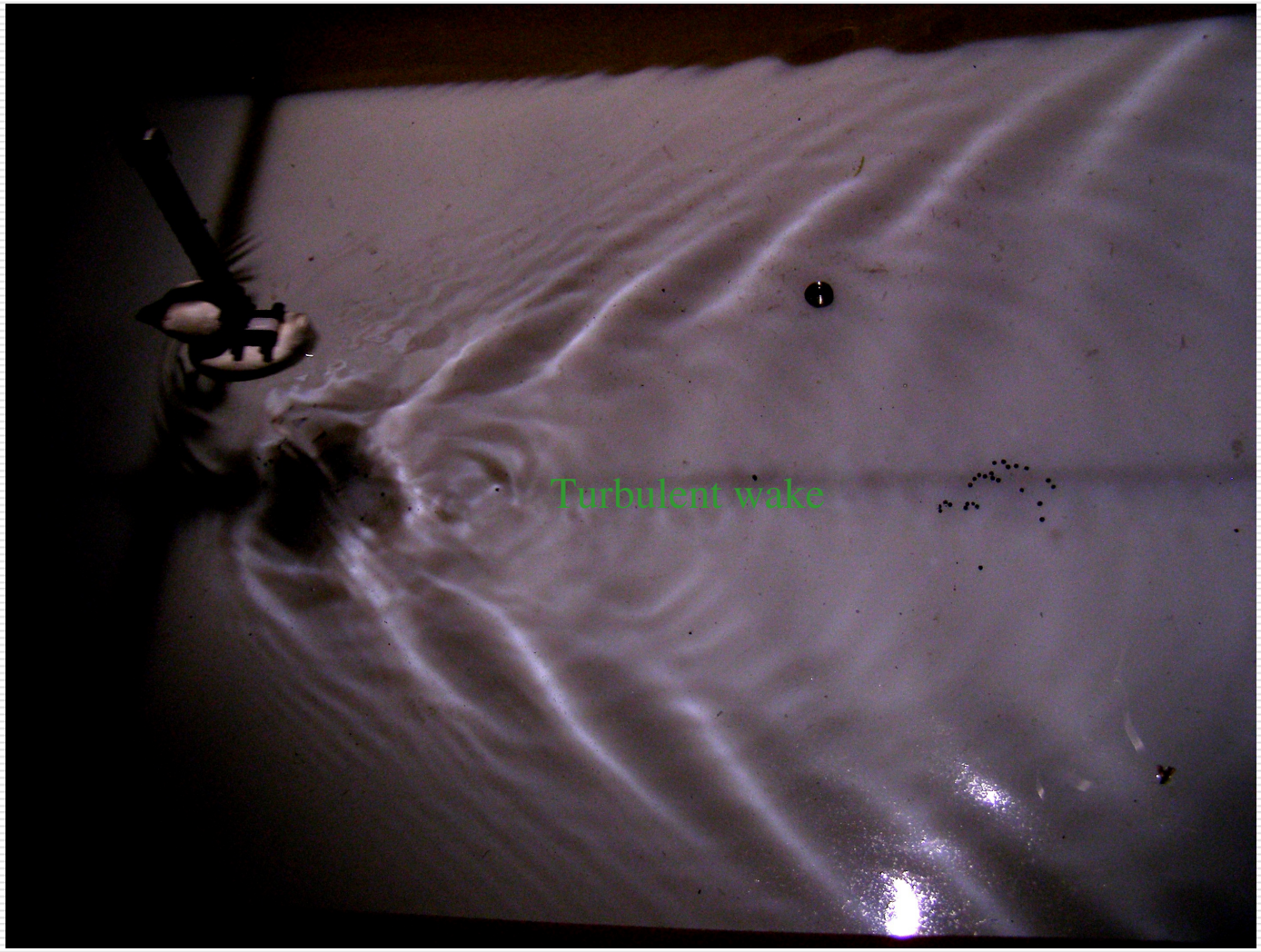
- Three forms used:
  - Duckling
  - Nail
  - Long form

Basic pattern properties cont.



- Large forms – two systems:
  - Bow wave
  - Stern wave
- Observed in real duck cones

Basic pattern properties cont.



- Oblique forms – smaller transverse waves
- Large turbulent wake
- Most realistic form

Advanced theory

- Target – obtaining real amplitude distribution on surface
- Base – wave equation in 2D
- Solution – transformed Fourier-Bessel integral
- Surface elevation – result of interference of all waves:

$$\eta(r) = \frac{v}{l} \int_{-\infty}^{0} \psi(r,t)\,dt$$

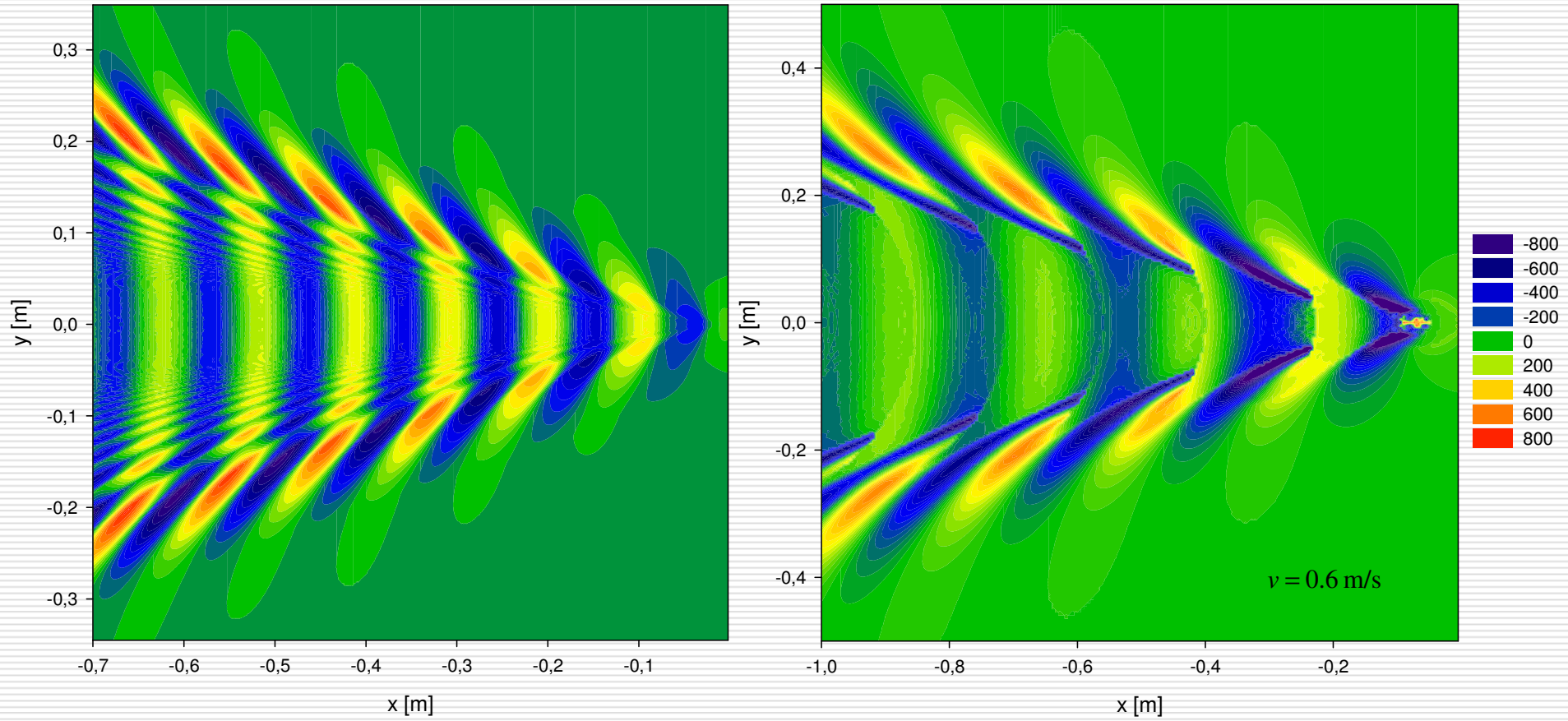
v – duck velocity

l – characteristic length

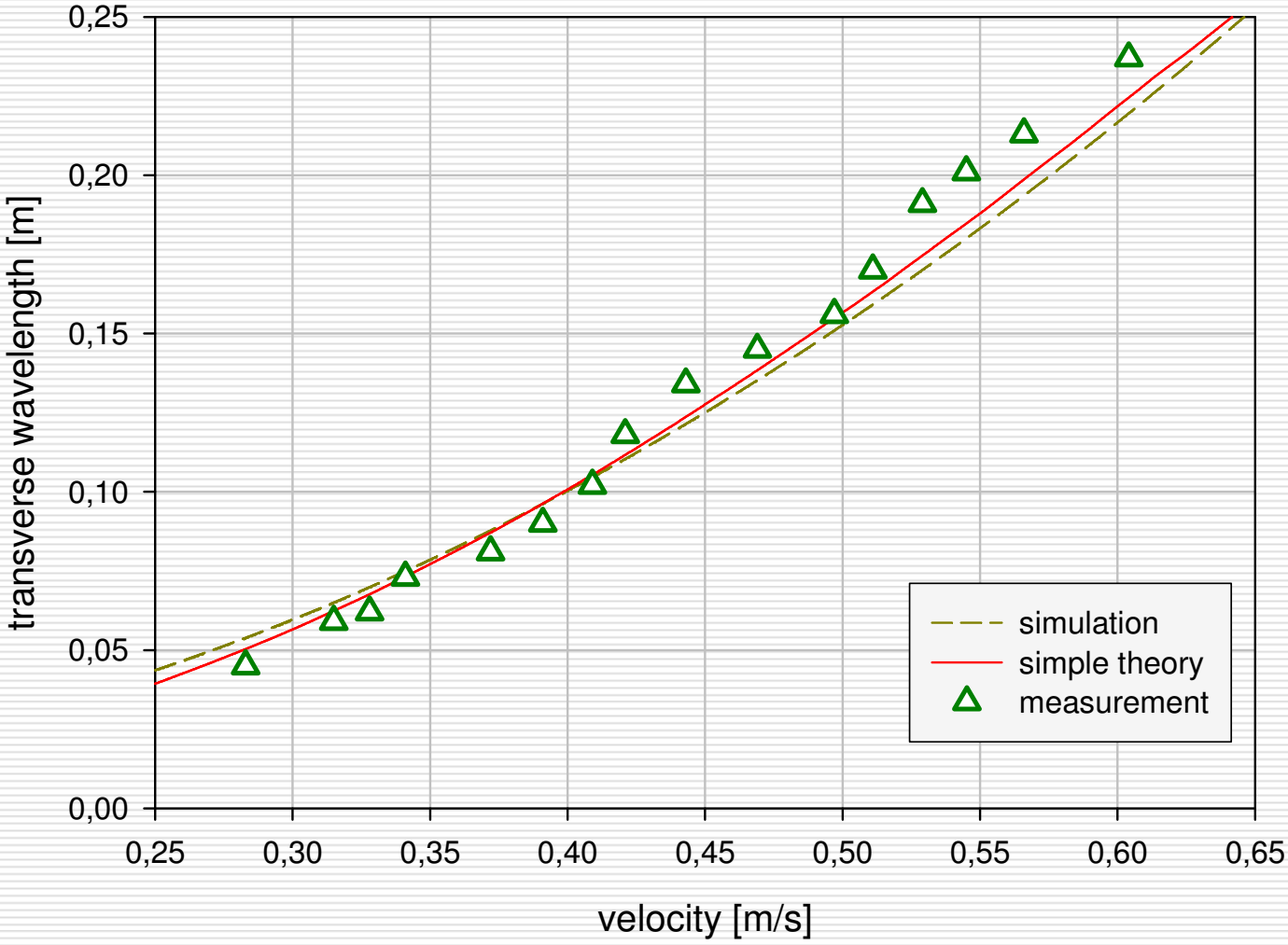
η – net surface elevation

Advanced theory cont.

- Numerical integration:



Comparison


0,25
0,20
0,15
0,10
0,05
0,00
transverse wavelength [m]
0,25
0,30
0,35
0,40
0,45
0,50
0,55
0,60
0,65
velocity [m/s]
simulation
simple theory
measurement

Comparison